Third rank Killing tensors in general relativity. The (1+1)-dimensional case.

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Abstract

Third rank Killing tensors in (1+1)-dimensional geometries are investigated and classified. It is found that a necessary and sufficient condition for such a geometry to admit a third rank Killing tensor can always be formulated as a quadratic PDE, of order three or lower, in a Kähler type potential for the metric. This is in contrast to the case of first and second rank Killing tensors for which the integrability condition is a linear PDE. The motivation for studying higher rank Killing tensors in (1+1)-geometries, is the fact that exact solutions of the Einstein equations are often associated with a first or second rank Killing tensor symmetry in the geodesic flow formulation of the dynamics. This is in particular true for the many models of interest for which this formulation is (1+1)-dimensional, where just one additional constant of motion suffices for complete integrability. We show that new exact solutions can be found by classifying geometries admitting higher rank Killing tensors.

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1   Introduction

Killing tensors are indispensable tools in the quest for exact solutions in many branches of general relativity as well as classical mechanics. For nontrivial examples where Killing vectors (*i.e.* first rank Killing tensors) and second rank Killing tensors have been used to find and classify solutions of the Einstein equations the reader is referred to [6, 9, 13] and references therein. However, there are no examples of exact solutions which correspond to third or higher rank Killing tensors. Killing tensors can also be important for solving the equations of motion in particular spacetimes. The notable example here is the Kerr metric which admits a second rank Killing tensor [5].

However, none of the classical exact solutions of the Einstein equations are known to admit higher rank Killing tensors. Recently, an example was given of a spacetime with a physically reasonable energy-momentum tensor admitting a third rank Killing tensor [8]. The method used in that work was based on Lax pair tensors [7], a concept which can be viewed as a generalization of Killing-Yano tensors [1]. In this paper we discuss the equations for third rank Killing tensors using a more direct approach in the spirit of [9] but modified to take into account the qualitative differences in the third rank case. Our emphasis will be on ideas and concepts and most of the results will be presented without proof. The reader who wishes to see more details can consult [3].

Any Killing tensor of rank two or higher has a traceless part which is itself a conformal Killing tensor. Furthermore the Killing tensor equations (for rank two or higher) can be decomposed in a traceless part and a trace part. The traceless part constitutes the conformal Killing tensor equations and involve only the traceless part of the Killing tensor. The trace part on the other hand involves both the trace and the traceless parts. In the second rank case the equation for the trace (which is then a scalar) gives rise to a covariant integrability condition involving only the conformal Killing tensor. In general such a covariant integrability condition is lacking for Killing tensors of rank three or higher. However, since the trace part of the Killing tensor equations in the third rank case is itself a second rank tensor its trace is a scalar equation. It turns out that this double trace equation is exactly the condition that the trace of the Killing tensor is divergence free. The third rank Killing tensor equations therefore decompose into three parts, one which involves only the conformal Killing tensor, one which involves only the trace vector and finally one part which couples the trace to the conformal Killing tensor.

In the present paper we focus on third rank Killing tensors in (1+1)-dimensional geometries. Such geometries are relevant to the study of solutions of the Einstein equations for such diverse areas as anisotropic cosmologies, inflationary cosmologies and relativistic star models [13]. Applying our approach to the (1+1)-dimensional case we are able to give a complete classification of the third rank Killing tensors. It turns out that any third rank conformal Killing tensor can be uniquely characterized by a real conformal Killing vector. This implies that there are two main types of third rank Killing tensors depending on whether the causal character of the conformal Killing vector is non-null or null. The classification is then refined by considering the scalar product of the conformal Killing vector with the trace vector. To solve the Killing tensor equations the first step is to observe that the divergence free property of the trace vector can be utilized to define a scalar potential for the trace vector. Using the scalar potential leads to simplification of the remaining Killing tensor equations. The form of those equations depends on the causal character of the conformal vector and on the scalar product of the conformal Killing vector with the trace vector. However, in all cases it is possible to find an integrability condition which involves only a Kähler type potential for the metric. Unlike the second rank case where the integrability conditions are linear, the third rank case leads to integrability conditions which are quadratic in the Kähler potential.

It has not been possible to find the general solution of the integrability conditions, except in the case where both the conformal Killing vector and the trace vector are null and have vanishing scalar product. However, we do give examples of solutions for all cases. We also consider the special case where the metric admits a homothetic Killing vector. In particular we give a complete treatment of the homothetic metrics with two exponential terms. It turns out that the only new integrable geometry in that case has complex exponential coefficients and therefore has a trigonometric potential. It is in fact a special case of a (1+1)-dimensional version of a 3-particle Toda lattice. Except for the homothetic case, most of the solutions given here represent new integrable (1+1)-dimensional geometries.

2 Third rank Killing tensors in (1+1)-dimensional spacetimes

Analogously to the second rank case investigated in [9] we shall make use of the fact that on any n-dimensional Riemannian or Lorentzian manifold, a third rank Killing tensor can be decomposed into its trace K_α and trace-free (conformal) part $P_{\alpha\beta\gamma}$ according to

$$K_{\alpha\beta\gamma} = P_{\alpha\beta\gamma} + \frac{3}{n+2} K_{(\alpha} g_{\beta\gamma)}. \tag{1}$$

This makes the Killing tensor equations $K_{(\alpha\beta\gamma;\delta)} = 0$ split into the conformal Killing tensor equations for the trace-free part,

$$C_{\alpha\beta\gamma\delta} := P_{(\alpha\beta\gamma;\delta)} - \frac{3}{n+4} g_{(\alpha\beta} P^\lambda{}_{\gamma\delta);\lambda} = 0, \tag{2}$$

and an equation which relates the trace-free part to the trace,

$$D_{\alpha\beta} := K_{(\alpha;\beta)} + \frac{n+2}{n+4} P^\gamma{}_{\alpha\beta;\gamma} = 0. \tag{3}$$

By taking the trace of eq. (3), one splits off the condition that K^α be divergence-free,

$$K^\alpha{}_{;\alpha} = 0. \tag{4}$$

Hence it is natural to start by solving the two decoupled conditions, eq. (2) and (4), before attempting to solve the remaining (i.e. trace-free) part of eq. (3). Focusing on the (1+1)-dimensional case, we use null variables and write the general metric as

$$ds^2 = -2G(u,\bar{u})dud\bar{u} = -2\Omega^0\Omega^1, \tag{5}$$

where we have introduced the standard null frame Ω^A, $A = 0, 1$, given by

$$\Omega^0 = G^{1/2}du, \quad \Omega^1 = G^{1/2}d\bar{u}. \tag{6}$$

We shall consistently use the convention that the two-dimensional tensor indices in this frame will take the values 0 and 1, while in a coordinate frame they take the values u and $\bar{u}$. To achieve maximal simplication of the Killing tensor equations we use the following parametrization of the Killing tensor (cf. the second rank case [9])

$$\begin{aligned}
K_{000} &= -RG^{3/2} \\
K_{111} &= -SG^{3/2} \\
K_{001} &= -\tfrac{1}{2}K_u G^{-1/2} \\
K_{011} &= -\tfrac{1}{2}K_{\bar{u}} G^{-1/2},
\end{aligned} \tag{7}$$

with $R := P^{\bar{u}\bar{u}\bar{u}}$ and $S := P^{uuu}$, using a notation analogous to the second rank case. The difference, which is solely due to the Killing tensor rank, being that R and S here are multiplied by $-G^{3/2}$ instead of G.

With the above parametrization, the conformal Killing tensor equations are simply

$$C_{0000} = -GR_{,u} = 0, \quad C_{1111} = -GS_{,\bar{u}} = 0, \tag{8}$$

requiring precisely that R and S be arbitrary functions of $\bar{u}$ and u respectively. This in fact shows that in two dimensions, any third rank conformal Killing tensor $P_{\alpha\beta\gamma}$ can in a unique way be represented by a conformal Killing *vector* ζ_α satisfying the equations

$$C_{\alpha\beta} := \zeta_{(\alpha;\beta)} - \frac{1}{n}\zeta^\gamma{}_{;\gamma} g_{\alpha\beta} = 0. \tag{9}$$

3

This can be shown as follows. In terms of the components ζ^u and $\zeta^{\bar{u}}$, the conformal Killing vector equations in the (1+1)-dimensional case reduce to

$$C_{00} = -\zeta^{\bar{u}},_u = 0, \quad C_{11} = -\zeta^u,_{\bar{u}} = 0. \tag{10}$$

These equations are solved by setting $\zeta^u = s(u)$, $\zeta^{\bar{u}} = r(\bar{u})$. The existence of such a large class of solutions reflects the fact that the conformal group in two dimensions is of (uncountably) infinite dimension. By choosing $s(u)$ and $r(\bar{u})$ appropriately, we can make our conformal Killing tensor $P_{\alpha\beta\gamma}$ become the trace-free part of $\zeta_\alpha \zeta_\beta \zeta_\gamma$, that is to say

$$E_{\alpha\beta\gamma} := \zeta_\alpha \zeta_\beta \zeta_\gamma - \frac{3}{4}\zeta^\delta \zeta_\delta \zeta_{(\alpha} g_{\beta\gamma)} - P_{\alpha\beta\gamma} = 0. \tag{11}$$

In component form, these equations become

$$E_{000} = -\{[r(\bar{u})]^3 - R(\bar{u})\}G^{3/2} = 0, \quad E_{111} = -\{[s(u)]^3 - S(u)\}G^{3/2} = 0. \tag{12}$$

Hence given any conformal Killing tensor $P_{\alpha\beta\gamma}$, there is as claimed a unique *real* conformal Killing vector ζ_α, given by $s(u) = [S(u)]^{1/3}$, $r(\bar{u}) = [R(\bar{u})]^{1/3}$ (the real cubic roots), which represents $P_{\alpha\beta\gamma}$ acoording to eq. (11). We shall use this result to characterize $P_{\alpha\beta\gamma}$ invariantly in terms of the causal character of ζ_α.

The divergence-free condition for K^α reads

$$K^\alpha,_\alpha = -G^{-1}(K_{u,\bar{u}} + K_{\bar{u},u}) = 0, \tag{13}$$

which we solve by setting $K_u = 2\Phi,_u$, $K_{\bar{u}} = -2\Phi,_{\bar{u}}$ for some arbitrary potential function $\Phi(u,\bar{u})$. This can be expressed covariantly in terms of the natural volume form $\epsilon_{\alpha\beta} = G(du \wedge d\bar{u})_{\alpha\beta}$ as

$$K_\alpha = 2\epsilon_\alpha{}^\beta \Phi_{;\beta}, \tag{14}$$

making eq. (3) take the form

$$D_{\alpha\beta} = 2\Phi_{;\gamma(\alpha}\epsilon_{\beta)}{}^\gamma + \frac{2}{3}P^\gamma{}_{\alpha\beta;\gamma} = 0, \tag{15}$$

The components of this equation, reading

$$D_{00} = 2\left(\frac{\Phi,_u}{G}\right),_u + \frac{2}{3}G^{-2}(G^3 R),_{\bar{u}} = 0, \tag{16}$$

$$D_{11} = -2\left(\frac{\Phi,_{\bar{u}}}{G}\right),_{\bar{u}} + \frac{2}{3}G^{-2}(G^3 S),_u = 0, \tag{17}$$

can be simplified by making a suitably chosen conformal transformation $u = F(U)$, $\bar{u} = \bar{F}(\bar{U})$ together with a corresponding frame scaling (boost) $\tilde{\Omega}^0 = B\Omega^0$, $\tilde{\Omega}^1 = B^{-1}\Omega^1$, which up to the trivial transformation $u \leftrightarrow \bar{u}$ will bring the Killing tensor to one of three inequivalent standard forms. Since the conformal factor transforms into $\tilde{G} = F'(U)\bar{F}'(\bar{U})G$, the new frame will be defined analogously to the old one, but in new null variables, by choosing $B = (\bar{F}'/F')^{1/2}$. We furthermore write the inverse of the conformal transformation as $U = H(u)$, $\bar{U} = \tilde{H}(\bar{u})$. Now, a Killing tensor is called reducible (and is thereby redundant for solving the geodesic equations) if it can be written as a linear combination of symmetrized tensor products of lower rank Killing tensors and the metric. Hence $K_{\alpha\beta\gamma}$ is automatically reducible if the conformal part $P_{\alpha\beta\gamma}$ is zero, since in that case the trace K_α is required to satisfy the Killing vector equation. Therefore we only take interest in the case when either $S(u)$ or $R(\bar{u})$ is nonzero. Moreover if $S(u)$ and $R(\bar{u})$ are both nonzero (i.e. if ζ_α is non-null) we can fix the conformal gauge along the same lines as in [9] by making a conformal transformation which sets $S(u)$ and $R(\bar{u})$ to the standard value 1. However, in the case when ζ_α is null so that either R or S is zero, this requirement only fixes one of the new variables U and $\bar{U}$. To fix the other variable we use two distinct conformal transformations depending on whether the scalar product $\zeta^\alpha K_\alpha$ vanishes or not. This makes it very natural to define three major types of third rank Killing tensors

4

corresponding to the three qualitatively different ways in which the conformal gauge is fixed. In table 1 this classification is summarized invariantly in terms of the scalars $\zeta^\alpha \zeta_\alpha$ and $\zeta^\alpha K_\alpha$.

For each Killing tensor type we shall now perform the conformal transformation and derive a necessary and sufficient integrability condition for eq. (15). When doing this it will be instructive to let the Killing tensor be represented in terms of the geodesic invariant $I := K^{\alpha\beta\gamma} p_\alpha p_\beta p_\gamma$, which has the general form

$$I = S p_u{}^3 + R p_u{}^3 + 3(-\Phi_{,\bar{u}} p_u + \Phi_{,u} p_{\bar{u}}) G^{-2} p_u p_{\bar{u}}. \tag{18}$$

Killing tensor type	$\zeta^\alpha \zeta_\alpha$	$\zeta^\alpha K_\alpha$	Nontriviality condition
I	$\neq 0$	no restriction	$K^\alpha K_\alpha \neq 0$, $\epsilon_{\alpha\beta} \zeta^\alpha K^\beta \neq 0$
IIA	0	$\neq 0$	$K^\alpha K_\alpha \neq 0$
IIB	0	0	

Table 1: Invariant classification of third rank Killing tensors in (1+1)-dimensional geometries.

Type I: $\zeta^\alpha \zeta_\alpha \neq 0$.

Since an arbitrary conformal transformation brings $S(u)$ and $R(\bar{u})$ into

$$\tilde{S}(U) := P^{UUU} = [H'(u)]^3 P^{uuu} = [H'(u)]^3 S(u), \tag{19}$$

$$\tilde{R}(\bar{U}) := P^{\bar{U}\bar{U}\bar{U}} = [\bar{H}'(\bar{u})]^3 P^{\bar{u}\bar{u}\bar{u}} = [\bar{H}'(\bar{u})]^3 R(\bar{u}), \tag{20}$$

we obtain $\tilde{S}(U) = 1$, $\tilde{R}(\bar{U}) = 1$, by choosing $H'(u) = [S(u)]^{-1/3}$, $\bar{H}'(\bar{u}) = [R(\bar{u})]^{-1/3}$. With this choice, eq. (16) and (17) are transformed into

$$\tilde{D}_{00} := B^{-2} D_{00} = 2\left[\left(\frac{\Phi_{,U}}{\tilde{G}}\right)_{,U} + \tilde{G}_{,\bar{U}}\right] = 0, \tag{21}$$

$$\tilde{D}_{11} := B^2 D_{11} = 2\left[-\left(\frac{\Phi_{,\bar{U}}}{\tilde{G}}\right)_{,\bar{U}} + \tilde{G}_{,U}\right] = 0. \tag{22}$$

Evidently, if $\Phi_{,U}$ (or $\Phi_{,\bar{U}}$) is zero, $\bar{U}$ (or U) is required to be a null cyclic variable in $\tilde{G}$, implying that the geometry is flat, and hence the case when ζ_α is non-null is interesting only when the trace K_α is non-null as well. In table 1 this is indicated as a nontriviality condition for type I.

The two equations (21) and (22) clearly have the integrability condition

$$\left(\frac{\Phi_{,U}}{\tilde{G}}\right)_{,UU} + \left(\frac{\Phi_{,\bar{U}}}{\tilde{G}}\right)_{,\bar{U}\bar{U}} = 0, \tag{23}$$

leading to

$$\Phi_{,U} = -\tilde{G} \mathcal{K}_{,\bar{U}\bar{U}}, \quad \Phi_{,\bar{U}} = \tilde{G} \mathcal{K}_{,UU} \tag{24}$$

for some potential function $\mathcal{K}$. Substituting this back into eq. (21) and (22) yields

$$\begin{aligned}\tilde{D}_{00} &= 2(-\mathcal{K}_{,U\bar{U}\bar{U}} + \tilde{G}_{,\bar{U}}) = 0, \\ \tilde{D}_{11} &= 2(-\mathcal{K}_{,UU\bar{U}} + \tilde{G}_{,U}) = 0,\end{aligned} \tag{25}$$

showing that since $\mathcal{K}$ is determined only up to $\mathcal{K} \to \mathcal{K} + \psi$ with $\psi_{,UU} = \psi_{,\bar{U}\bar{U}} = 0$, it can be chosen such that

$$\tilde{G} = \mathcal{K}_{,U\bar{U}} \tag{26}$$

5

holds. Identifying our null variables with complex conjugate variables and borrowing terminology from the theory of complex manifolds (see e.g. [4]), the relation (26) shows that $\mathcal{K}$ plays the role of a Kähler potential for the metric. A Kähler potential $\mathcal{K}$ has the property of transforming as a scalar under conformal transformations since $G = \mathcal{K}_{,u\bar{u}}$ clearly implies $\tilde{G} = \mathcal{K}_{,U\bar{U}}$. On the other hand it has the disadvantage of being determined only up to a gauge transformation $\mathcal{K} \to \mathcal{K} + f(u) + g(\bar{u})$. Requiring that $\mathcal{K}$ should satisfy (24) as well as (26) however fixes the gauge up to addition of a linear function of U and $\bar{U}$. Substituting eq. (26) into eq. (24) leads immediately to the standardized integrability condition

$$(\mathcal{K}_{,U\bar{U}}\mathcal{K}_{,UU})_{,U} + (\mathcal{K}_{,U\bar{U}}\mathcal{K}_{,\bar{U}\bar{U}})_{,\bar{U}} = 0, \tag{27}$$

which is necessary and sufficient for the existence of a third rank Killing tensor of this type. Transforming back to the arbitrary null variables u and $\bar{u}$ (without changing the Kähler gauge), eq. (24) becomes

$$\Phi_{,u} = -\mathcal{K}_{,u\bar{u}}(R\mathcal{K}_{,\bar{u}\bar{u}} + \frac{1}{3}R'\mathcal{K}_{,\bar{u}}), \quad \Phi_{,\bar{u}} = \mathcal{K}_{,u\bar{u}}(S\mathcal{K}_{,uu} + \frac{1}{3}S'\mathcal{K}_{,u}), \tag{28}$$

with the corresponding integrability condition

$$[\mathcal{K}_{,u\bar{u}}(3S\mathcal{K}_{,uu} + S'\mathcal{K}_{,u})]_{,u} + [\mathcal{K}_{,u\bar{u}}(3R\mathcal{K}_{,\bar{u}\bar{u}} + R'\mathcal{K}_{,\bar{u}})]_{,\bar{u}} = 0. \tag{29}$$

Since the Killing tensor building blocks $P_{\alpha\beta\gamma}$, K_{α} and $g_{\alpha\beta}$ all have been expressed in terms of $S(u)$, $R(\bar{u})$ and $\mathcal{K}$, we have the following closed expression for the geodesic invariant:

$$I = Sp_u{}^3 + Rp_{\bar{u}}{}^3 - [(3S\mathcal{K}_{,uu} + S'\mathcal{K}_{,u})p_u + (3R\mathcal{K}_{,\bar{u}\bar{u}} + R'\mathcal{K}_{,\bar{u}})p_{\bar{u}}]\mathcal{K}_{,u\bar{u}}{}^{-1}p_u p_{\bar{u}}, \tag{30}$$

which in the standardized null variables simplifies to

$$I = p_U{}^3 + p_{\bar{U}}{}^3 - 3(\mathcal{K}_{,UU}\,p_U + \mathcal{K}_{,\bar{U}\bar{U}}\,p_{\bar{U}})\mathcal{K}_{,U\bar{U}}{}^{-1}p_U p_{\bar{U}}. \tag{31}$$

Type II : $\zeta^{\alpha}\zeta_{\alpha} = 0$.

Since the case $S(u) = 0$ can be obtained from the case $R(\bar{u}) = 0$ by making the transformation $u \leftrightarrow \bar{u}$, we here only need to consider the case $S(u) \neq 0$, $R(\bar{u}) = 0$. We then solve eq. (16) immediately by introducing a function $Q(\bar{u})$ defined by the equation

$$\Phi_{,u} = Q(\bar{u})G. \tag{32}$$

The choice of the transformation function $\bar{H}(\bar{u})$ will now depend on whether $Q(\bar{u})$ is zero or non-zero, i.e. whether the scalar product $\zeta^{\alpha}K_{\alpha} = 2(S^{1/3}\Phi_{,u} - R^{1/3}\Phi_{,\bar{u}})$ vanishes or not. However, just as for type I we choose $H'(u) = [S(u)]^{-1/3}$ to obtain $\tilde{S}(U) = 1$, which regardless of $\bar{H}(\bar{u})$ makes eq. (17) transform into

$$\tilde{D}_{11} = 2\,[-(\frac{\Phi_{,\bar{U}}}{\tilde{G}})_{,\bar{U}} + \tilde{G}_{,U}] = 0. \tag{33}$$

As for type I, $\Phi_{,\bar{U}} = 0$ obviously leads to U being a null cyclic variable in $\tilde{G}$ which implies a flat geometry. In particular, this means that a Killing tensor of type IIA for which ζ_{α} is null and $\zeta^{\alpha}K_{\alpha} \neq 0$, can be nontrivial only when the trace K_{α} is non-null. This is indicated in table 1 as a nontriviality condition for type IIA.

Type IIA : $\zeta^{\alpha}\zeta_{\alpha} = 0$, $\zeta^{\alpha}K_{\alpha} \neq 0$.

Since $Q(\bar{u})$ has the transformation property

$$\tilde{Q}(\bar{U}) := \Phi_{,U}/\tilde{G} = \bar{H}'(\bar{u})\Phi_{,u}/G = \bar{H}'(\bar{u})Q(\bar{u}) \tag{34}$$

and here is nonzero, it is clear that the conformal gauge can be fixed by choosing $\bar{H}'(\bar{u}) = Q(\bar{u})^{-1}$ which makes $\tilde{Q}(\bar{U})$ take the standard value 1.

Working in the standardized null variables, we now substitute $\tilde{G} = \Phi_{,U}$ into eq. (17) to yield the final nonlinear condition

$$\tilde{D}_{11} = 2[-(\frac{\Phi_{,\bar{U}}}{\Phi_{,U}})_{,\bar{U}} + \Phi_{,UU}] = 0, \tag{35}$$

which in expanded form reads

$$\Phi_{,U}{}^2\Phi_{,UU} + \Phi_{,\bar{U}}\Phi_{,U\bar{U}} - \Phi_{,U}\Phi_{,\bar{U}\bar{U}} = 0. \tag{36}$$

Transforming back to arbitrary null variables, this condition becomes

$$\Phi_{,u}{}^2(S\Phi_{,uu} + \frac{1}{3}S'\Phi_{,u}) + Q^2(\Phi_{,\bar{u}}\Phi_{,u\bar{u}} - \Phi_{,u}\Phi_{,\bar{u}\bar{u}}) - QQ'\Phi_{,u}\Phi_{,\bar{u}} = 0. \tag{37}$$

Using now the fact that we have expressed $P_{\alpha\beta\gamma}$, K_α and $g_{\alpha\beta}$ in terms of the functions $S(u)$, $Q(\bar{u})$ and Φ, the geodesic invariant takes the closed form

$$I = Sp_u{}^3 + 3(-\Phi_{,\bar{u}}p_u + \Phi_{,u}p_{\bar{u}})Q^2\Phi_{,u}{}^{-2}p_u p_{\bar{u}} \tag{38}$$

in arbitrary null variables, reducing to

$$I = p_U{}^3 + 3(-\Phi_{,\bar{U}}p_U + \Phi_{,U}p_{\bar{U}})\Phi_{,U}{}^{-2}p_U p_{\bar{U}} \tag{39}$$

in standardized null variables. If we let the metric be given by a Kähler potential $\mathcal{K}$ as $\tilde{G} = \mathcal{K}_{,U\bar{U}}$, the relation $\tilde{G} = \Phi_{,U}$ shows that it is possible to make a gauge transformation $\mathcal{K} \to \mathcal{K} + g(\bar{U})$ so that $\mathcal{K}_{,\bar{U}} = \Phi$ holds. Thus via eq. (35), $\mathcal{K}$ is required to satisfy

$$-(\frac{\mathcal{K}_{,\bar{U}\bar{U}}}{\mathcal{K}_{,U\bar{U}}})_{,\bar{U}} + \mathcal{K}_{,UU\bar{U}} = 0, \tag{40}$$

leading directly to

$$\mathcal{K}_{,U\bar{U}}\mathcal{K}_{,UU} - \mathcal{K}_{,\bar{U}\bar{U}} = h(U)\mathcal{K}_{,U\bar{U}} \tag{41}$$

for an arbitrary function $h(U)$. Using the remaining gauge freedom for $\mathcal{K}$, the transformation $\mathcal{K} \to \mathcal{K} + f(U)$ with $f''(U) = h(U)$ makes eq. (41) reduce to the standardized form

$$\mathcal{K}_{,U\bar{U}}\mathcal{K}_{,UU} = \mathcal{K}_{,\bar{U}\bar{U}}, \tag{42}$$

which corresponds to eq. (27) for type I. Transforming back to arbitrary null variables, the condition becomes

$$\mathcal{K}_{,u\bar{u}}(S\mathcal{K}_{,uu} + \frac{1}{3}S'\mathcal{K}_{,u}) = (Q\mathcal{K}_{,\bar{u}})_{,\bar{u}}. \tag{43}$$

Finally we express the geodesic invariant in alternative form in terms of the gauge fixed Kähler potential instead of the trace potential Φ:

$$\begin{aligned} I &= Sp_u{}^3 - 3[(S\mathcal{K}_{,uu} + \frac{1}{3}S'\mathcal{K}_{,u})p_u - Qp_{\bar{u}}]\mathcal{K}_{,u\bar{u}}{}^{-1}p_u p_{\bar{u}} \quad \text{(arbitrary null variables)}, \\ &= p_U{}^3 - 3(\mathcal{K}_{,UU}\,p_U - p_{\bar{U}})\mathcal{K}_{,U\bar{U}}{}^{-1}p_U p_{\bar{U}} \quad \text{(standardized null variables).} \end{aligned} \tag{44}$$

Type IIB : $\zeta^\alpha\zeta_\alpha = 0$, $\zeta^\alpha K_\alpha = 0$.

Since $Q(\bar{u})$ here vanishes, the scalar potential Φ is according to eq. (32) a function of $\bar{u}$ only and we can hence introduce the function $P(\bar{u}) := \Phi_{,\bar{u}}$, which transforms according to

$$\tilde{P}(\bar{U}) := \Phi_{,\bar{U}} = [\bar{H}'(\bar{u})]^{-1}\Phi_{,\bar{u}} = [\bar{H}'(\bar{u})]^{-1}P(\bar{u}). \tag{45}$$

If $P(\bar{u})$ vanishes, then so does the trace K_α. Since ζ_α is null, this would mean that

$$K_{\alpha\beta\gamma} = \zeta_\alpha\zeta_\beta\zeta_\gamma \tag{46}$$

which implies that ζ_α would be a null Killing vector. Disregarding this trivial case, we see that the conformal gauge can be fixed by making the choice $\bar{H}(\bar{u}) = P(\bar{u})$ in order to obtain $\tilde{P}(\bar{U}) = 1$.

Substituting $\Phi_{,\bar{U}} = 1$ into eq. (33) yields

$$\tilde{D}_{11} = 2[-(\tilde{G}^{-1})_{,\bar{U}} + \tilde{G}_{,U}] = 0, \tag{47}$$

that is

$$\tilde{G}^2\tilde{G}_{,U} + \tilde{G}_{,\bar{U}} = 0, \tag{48}$$

which is a quasi-linear first order equation in the conformal factor $\tilde{G}$. In arbitrary null variables this equation takes the form

$$G^2(SG_{,u} + \frac{1}{3}S'G) + PG_{,\bar{u}} - P'G = 0. \tag{49}$$

The geodesic invariant can here be directly expressed in terms of $S(u)$, $P(\bar{u})$ and the conformal factor G as

$$\begin{aligned} I &= Sp_u{}^3 - 3PG^{-2}p_u{}^2p_{\bar{u}} \quad \text{(arbitrary null variables)}, \\ &= p_U{}^3 - 3\tilde{G}^{-2}p_U{}^2p_{\bar{U}} \quad \text{(standardized null variables)}. \end{aligned} \tag{50}$$

To obtain a condition corresponding to eq. (27) for type I and eq. (42) for type IIA, we substitute $\tilde{G} = \mathcal{K}_{,U\bar{U}}$ into eq. (47) to obtain

$$-\mathcal{K}_{,U\bar{U}}{}^{-1} + \mathcal{K}_{,UU} = h(U), \tag{51}$$

for an arbitrary function $h(U)$. Standardizing the condition, we let $\mathcal{K} \to \mathcal{K} + f(U)$ with $f''(U) = h(U)$ yielding

$$\mathcal{K}_{,U\bar{U}}\mathcal{K}_{,UU} = 1, \tag{52}$$

or, in arbitrary null variables,

$$\mathcal{K}_{,u\bar{u}}(S\mathcal{K}_{,uu} + \frac{1}{3}S'\mathcal{K}_{,u}) = P. \tag{53}$$

As for type IIA, we now have an alternative expression for the geodesic invariant, namely

$$\begin{aligned} I &= Sp_u{}^3 - (3S\mathcal{K}_{,uu} + S'\mathcal{K}_{,u})\mathcal{K}_{,u\bar{u}}{}^{-1}p_u{}^2p_{\bar{u}} \quad \text{(arbitrary null variables)}, \\ &= p_U{}^3 - 3\mathcal{K}_{,UU}\mathcal{K}_{,U\bar{U}}{}^{-1}p_U{}^2p_{\bar{U}} \quad \text{(standardized null variables)}. \end{aligned} \tag{54}$$

Comment on reducibility

The number of independent Killing vectors can for a 2-dimensional geometry be three, one or zero. The highly symmetric geometries that admit three Killing vectors are precisely the ones that have constant scalar curvature. Such geometries cannot have higher order invariants that are independent of the three linear invariants. For geometries with precisely one Killing vector ξ_α but no irreducible second rank Killing tensors, a reducible third rank Killing tensor can only be of the form

$$K_{\alpha\beta\gamma} = C_1\xi_\alpha\xi_\beta\xi_\gamma + C_2\xi_{(\alpha}g_{\beta\gamma)} \tag{55}$$

for some constants C_1 and C_2. The Killing vector ξ_α is by necessity non-null, since the geometry otherwise would be flat. As we are not considering the automatically reducible case when the trace-free part of a third

rank Killing tensor vanishes, we assume that $C_1 \neq 0$ and redefine $K_{\alpha\beta\gamma}$ or ξ_α so that $C_1 = 1$. It then follows that the conformal Killing vector ζ_α coincides with ξ_α and that the trace K_α is related to ζ_α by

$$K_\alpha = (\zeta^\beta \zeta_\beta + \frac{4}{3} C_2) \zeta_\alpha. \tag{56}$$

In particular, since ζ_α is non-null, this reducible Killing tensor is of type I. It would be practical to have an invariant criterion which isolates this reducible case from the family of type I Killing tensors since it cannot be identified by checking if the curvature is constant. In fact such a criterion does exist. Noting that a necessary and sufficient condition according to eq. (56) is that ζ_α and K_α be parallell,

$$\epsilon_{\alpha\beta} \zeta^\alpha K^\beta = 2\zeta^\alpha \Phi_{;\alpha} = 0, \tag{57}$$

we will now show that it is also a sufficient condition. This means that given a third rank Killing tensor of type I, we must show that eq. (57) implies that ζ_α is a Killing vector and that eq. (56) holds. Now, in the standard variables for type I we have

$$\zeta^\alpha = (\partial/\partial U + \partial/\partial \bar{U})^\alpha, \tag{58}$$

so according to eq. (24) and (57), imposing that ζ_α and K_α be parallell implies that

$$\zeta^\alpha \Phi_{;\alpha} = \Phi_{,U} + \Phi_{,\bar{U}} = \tilde{G}(-\mathcal{K}_{,\bar{U}\bar{U}} + \mathcal{K}_{,UU}) = 0, \tag{59}$$

leading to $\mathcal{K} = f(U + \bar{U}) + g(U - \bar{U})$. Substituting this into the general integrability condition (27) gives the further restriction $f'''(U + \bar{U}) = 0$, i.e., up to irrelevant linear terms in $\mathcal{K}$,

$$\begin{aligned}
\mathcal{K} &= \tfrac{1}{2} A(U + \bar{U})^2 + g(U - \bar{U}) \\
\tilde{G} &= A - g''(U - \bar{U})
\end{aligned} \tag{60}$$

for some arbitrary constant A and function $g(U - \bar{U})$. Clearly, this shows that ζ_α is a Killing vector. Furthermore eq. (24) now implies that K_α can be written as $K_\alpha = (\zeta^\beta \zeta_\beta + 4A)\zeta_\alpha$. Comparing this with eq. (56) and reading off that $C_2 = 3A$ proves the assertion.

For geometries with an irreducible second rank Killing tensor, the situation is different. If there are no Killing vectors, there are no ways to construct a reducible third rank Killing tensor. This is of course not the case if a Killing vector does exist, but for such geometries we do not know of a simple invariant criterion which can be used to check if a third rank Killing tensor is irreducible.

To summarize, except for geometries which admit an irreducible second rank Killing tensor and precisely one Killing vector, irreducibility of a third rank Killing tensor is guaranteed if the geometry does not have constant curvature and, for type I, if $\epsilon_{\alpha\beta} \zeta^\alpha K^\beta \neq 0$.

3 Some Solutions to the Standardized Integrability Conditions

In this section we adress the problem of finding solutions to the final integrability conditions expressed in the adapted null variables $U, \bar{U}$. Due to the fact that these conditions are nonlinear PDE's, in contrast to the corresponding conditions for the existence of second rank Killing tensors [9], we shall have to settle for giving some examples of nontrivial solutions, rather then giving the general solutions. The exception is type IIB where the general solution for the conformal factor can be given in implicit form.

Type I

Let us begin by a remark on the symmetries of eq. (27). Obviously, the equation is invariant under coordinate translations $U \to U + U_0$, $\bar{U} \to \bar{U} + \bar{U}_0$ as well as under coordinate scalings $U \to cU$, $\bar{U} \to c\bar{U}$ and scalings of the dependent variable $\mathcal{K}$. Moreover, the equation has a discrete $Z_3 \times Z_3$ symmetry of being invariant

under $U \to e^{i2\pi m/3}U$, $\bar{U} \to e^{-i2\pi n/3}\bar{U}$ with m, $n = 0$, ± 1. When writing down explicit solutions below, we give only one representative in each of these symmetry gauge classes.

Due to the scaling symmetries of eq. (27), it is natural to make the ansatz that $\mathcal{K}$ is a homogeneous function of U and $\bar{U}$, i.e.

$$\mathcal{K}(cU, c\bar{U}) = c^\lambda \mathcal{K}(U, \bar{U}). \tag{61}$$

This implies that one can write $\mathcal{K} = U^\lambda f(\eta)$ with $\eta = \bar{U}/U$, which substituted into eq. (27) yields a complicated third order ODE for the function $f(\eta)$. For two values of λ, namely $\lambda = 1$ and $\lambda = 2$, it is possible to find the general solution to this equation. The solutions for these two cases read

$$\begin{cases} \mathcal{K} = -U \int\int \eta^{-1/2}(\eta^3 - 1)^{-2/3} d\eta d\eta = \int\int \sqrt{U\bar{U}}(U^3 - \bar{U}^3)^{-2/3} dU d\bar{U} \\ \tilde{G} = \sqrt{U\bar{U}}(U^3 - \bar{U}^3)^{-2/3}, \end{cases} \tag{62}$$

$$\begin{cases} \mathcal{K} = U^2[A(\eta^{3/2} + 1)^{4/3} - B(\eta^{3/2} - 1)^{4/3}] = A(U^{3/2} + \bar{U}^{3/2})^{4/3} - B(U^{3/2} - \bar{U}^{3/2})^{4/3}, \\ \tilde{G} = \sqrt{U\bar{U}}[A(U^{3/2} + \bar{U}^{3/2})^{-2/3} + B(U^{3/2} - \bar{U}^{3/2})^{-2/3}]. \end{cases} \tag{63}$$

The geometry corresponding to the solution (63) is here found to be superintegrable since it also admits a second rank non-null Killing tensor. This can be shown by transforming into new null variables $u = U^{3/2}$, $\bar{u} = \bar{U}^{3/2}$ after which the conformal factor will satisfy the wave equation $G_{,uu} = G_{,\bar{u}\bar{u}}$ [9]. Furthermore, for $\lambda = 3$ one has the special solution

$$\begin{cases} \mathcal{K} = \frac{4}{9}U^3\eta^3 = \frac{4}{9}(U\bar{U})^{3/2} \\ \tilde{G} = \sqrt{U\bar{U}}, \end{cases} \tag{64}$$

which is trivial since it corresponds to a flat geometry. However, an arbitrary linear combination of this solution and the solution (62) also solves eq. (27) and thus gives a nontrivial generalization of the latter case. A lesson to be learnt from this is that one should not reject homogeneous solutions which are trivial as they stand since they are potential building blocks for nontrivial inhomogeneous solutions.

Introducing non-null variables T and X defined by $U = T + X$, $\bar{U} = T - X$ in terms of which

$$\tilde{G} = \frac{1}{4}(\mathcal{K}_{,TT} - \mathcal{K}_{,XX}), \tag{65}$$

one easily verifies that eq. (27) is solved by letting $\mathcal{K}$ be an arbitrary function of $X = (U - \bar{U})/2$ only, which corresponds to ζ_α being a Killing vector. Consequently, the equation also has the complex solutions when $\mathcal{K}$ is a function of $X \pm i\sqrt{3}T = -e^{\mp i2\pi/3}U + e^{\pm i2\pi/3}\bar{U}$ only. These are of course trivial solutions by themselves, but they suggest that the ansatz

$$\mathcal{K} = f(-2X) + g(X + i\sqrt{3}T) + h(X - i\sqrt{3}T) \tag{66}$$

be made, for the simple reason that each term by itself satisfies the equation. Moreover, we shall assume that the three functions have the same functional dependence, i.e. that $f(z) = g(z) = h(z)$, thus ensuring that $\mathcal{K}$ is real and invariant under the Z_3 symmetry $U \to e^{i2\pi n/3}U$, $\bar{U} \to e^{-i2\pi n/3}\bar{U}$. Some special solutions obtained with this ansatz are

$$\begin{cases} f(z) = -e^z - \frac{1}{6}Az^2 \\ \tilde{G} = e^{-2X} + e^{X+i\sqrt{3}T} + e^{X-i\sqrt{3}T} + A = e^{-2X} + 2e^X \cos\sqrt{3}T + A, \end{cases} \tag{67}$$

$$\begin{cases} f(z) = \ln z - \frac{1}{6}Az^2 \\ \tilde{G} = (-2X)^{-2} + (X + i\sqrt{3}T)^{-2} + (X - i\sqrt{3}T)^{-2} + A \\ \quad = \dfrac{9(X^2 - T^2)^2}{4X^2(X^2 + 3T^2)^2} + A, \end{cases} \tag{68}$$

$$
\begin{cases}
f(z) = \ln(\sinh z) - \frac{1}{6}Az^2 \\
\tilde{G} = [\sinh(-2X)]^{-2} + [\sinh(X + i\sqrt{3}T)]^{-2} + [\sinh(X - i\sqrt{3}T)]^{-2} + A \\
\quad = 2\, \dfrac{\cos 4\sqrt{3}T + 2(\cosh 6X - 3\cosh 2X)\cos 2\sqrt{3}T - 3\cosh 4X + 6}{(2\sinh 2X \cos 2\sqrt{3}T - \sinh 4X)^2} + A,
\end{cases}
\tag{69}
$$

$$
\begin{cases}
f(z) = \ln(\sin z) - \frac{1}{6}Az^2 \\
\tilde{G} = [\sin(-2X)]^{-2} + [\sin(X + i\sqrt{3}T)]^{-2} + [\sin(X - i\sqrt{3}T)]^{-2} + A \\
\quad = 2\, \dfrac{\cosh 4\sqrt{3}T + 2(\cos 6X - 3\cos 2X)\cosh 2\sqrt{3}T - 3\cos 4X + 6}{(2\sin 2X \cosh 2\sqrt{3}T - \sin 4X)^2} + A.
\end{cases}
\tag{70}
$$

All of the solutions (67) - (70) correspond to well-known classical mechanical potentials that are integrable with a cubic invariant [2]. In particular, the conformal factor for the first solution (67) is the Lorentzian analogue of the three-particle Toda potential. This solution differs from the others obtained with the given ansatz in that its three exponential terms can have arbitrary constant coefficients, which means for this case it is not necessary that $\mathcal{K}$ obeys the Z_3 symmetry. In the case when the arbitrary constant A is zero, the metric corresponding to the solution (68) has constant but nonzero curvature and then admits three independent non-null Killing vectors.

When setting $\mathcal{K}$ to an arbitrary function of T only, eq. (27) requires that this function be a second degree polynomial. A natural ansatz is therefore obtained by replacing the polynomial coefficients with arbitrary functions of X, i.e.

$$
\mathcal{K} = f(X)T^2 + g(X)T + h(X).
\tag{71}
$$

The general solution with this ansatz reads

$$
\begin{cases}
\mathcal{K} = (-9AX^{4/3} + 4D)T^2 - 9BX^{4/3}T - \frac{27}{14}AX^{10/3} - 9CX^{4/3} + 2DX^2 \\
\tilde{G} = -\frac{3}{4}AX^{4/3} + (AT^2 + BT + C)X^{-2/3} + D,
\end{cases}
\tag{72}
$$

with some irrelevant integration constants set to zero. In the case when $A \neq 0$, the solution can be further standardized by setting $A = 1$, $B = 0$. It can then be identified as the Lorentzian analogue of Holt's integrable classical mechanical potential [2]. If $A = 0$ but $B \neq 0$, we still have a nontrivial solution which is standardized by setting $B = 1$, $C = 0$.

We have seen that imposing that ζ_α and K_α be parallell leads to ζ_α being a Killing vector in terms of which $K_{\alpha\beta\gamma}$ is reducible. We here instead make the ansatz that ζ_α and K_α be orthogonal,

$$
\zeta^\alpha K_\alpha = 2(\Phi_{,U} - \Phi_{,\bar{U}}) = -2\mathcal{K}_{,U\bar{U}}(\mathcal{K}_{,\bar{U}\bar{U}} + \mathcal{K}_{,UU}) = 0.
\tag{73}
$$

This leads directly to $\mathcal{K}$ being a harmonic function of U and $\bar{U}$, i.e.

$$
\mathcal{K} = f(U + i\bar{U}) + \bar{f}(U - i\bar{U}),
\tag{74}
$$

where f is an analytic function of $U + i\bar{U}$. When substituting eq. (74) into eq. (27) one obtains the nontrivial solution

$$
\begin{cases}
\mathcal{K} = -\frac{2}{15}\{[(1 + i)(U + i\bar{U})]^{5/2} + [(1 - i)(U - i\bar{U})]^{5/2}\} \\
\tilde{G} = \sqrt{(1 + i)(U + i\bar{U})} + \sqrt{(1 - i)(U - i\bar{U})} = \sqrt{2[\sqrt{2(U^2 + \bar{U}^2)} + U - \bar{U}]}.
\end{cases}
\tag{75}
$$

Since the conformal factor satisfies the Laplace equation $\tilde{G}_{,UU} + \tilde{G}_{,\bar{U}\bar{U}} = 0$, the geometry also admits a non-null second rank Killing tensor [9] and is thus superintegrable.

Type IIA

We shall here give the results in terms of the trace potential Φ, which in this case also serves as a potential for the metric via the relation $\tilde{G} = \Phi_{,U}$. Contrary to the Kähler potential condition (42), the equivalent condition (36) imposed on Φ is a PDE which is linear in the second derivatives. However, this does not by necessity mean that eq. (36) in general is easier to work with than eq. (42), since the latter has the advantage of being quadratic instead of cubic in the dependent function, besides being a more compact equation.

Noting that eq. (36) is invariant under translations $U \to U + U_0$, $\bar{U} \to \bar{U} + \bar{U}_0$ as well as the correlated scalings $U \to aU$, $\bar{U} \to b\bar{U}$, $\Phi \to a^{-3}b^2\Phi$ (that is to say, if $\Phi = f(U, \bar{U})$ is a solution, then so is $\Phi = a^{-3}b^2 f(aU, b\bar{U})$), the solutions can preferably be exhibited with the freedom to make these transformations fixed, but it is often convenient to avoid a complete fixing in order to be able to let several inequivalent subcases be contained in one single expression.

By trial and error, one quickly finds that the two ansätze

$$\Phi = f(\bar{U})U^3 + g(\bar{U})U^2 + h(\bar{U})U + k(\bar{U}), \tag{76}$$

$$\Phi = f(\bar{U})U^3 + g(\bar{U})U^{3/2} + h(\bar{U}), \tag{77}$$

give rise to nontrivial solutions obtained by solving ODE's for the coefficient functions. The following solutions have been found:

$$
\begin{cases}
\Phi = \frac{1}{9}\bar{U}^{-2}U^3 + (A\bar{U}^{-2} + B\bar{U}^{-1/3})U^2 + [3(A\bar{U}^{-1} + B\bar{U}^{2/3})^2 + C\bar{U}^{-2/3}]U \\
\qquad + 3(A\bar{U}^{-2/3} + B\bar{U})^3 + 3C(A\bar{U}^{-2/3} + B\bar{U}) \\
\tilde{G} = \frac{1}{3}\bar{U}^{-2}U^2 + 2(A\bar{U}^{-2} + B\bar{U}^{-1/3})U + 3(A\bar{U}^{-1} + B\bar{U}^{2/3})^2 + C\bar{U}^{-2/3}
\end{cases} \tag{78}
$$

$$
\begin{cases}
\Phi = -\frac{1}{9}(\cosh\bar{U})^{-2}U^3 + A(\cosh\bar{U})^{-2/3}U \\
\tilde{G} = -\frac{1}{3}(\cosh\bar{U})^{-2}U^2 + A(\cosh\bar{U})^{-2/3}
\end{cases} \tag{79}
$$

$$
\begin{cases}
\Phi = \frac{1}{9}(\cos\bar{U})^{-2}U^3 + A(\cos\bar{U})^{-2/3}U \\
\tilde{G} = \frac{1}{3}(\cos\bar{U})^{-2}U^2 + A(\cos\bar{U})^{-2/3}
\end{cases} \tag{80}
$$

$$
\begin{cases}
\Phi = \frac{1}{2}AU^2 + (\frac{1}{2}A^2\bar{U}^2 + B\bar{U} + C)U \\
\qquad + \frac{1}{8}A^3\bar{U}^4 + \frac{1}{2}AB\bar{U}^3 + \frac{1}{2}(A^{-1}B^2 + AC)\bar{U}^2 + A^{-1}BC\bar{U} \\
\tilde{G} = AU + \frac{1}{2}A^2\bar{U}^2 + B\bar{U} + C
\end{cases} \tag{81}
$$

$$
\begin{cases}
\Phi = \frac{1}{2}Ae^{2\bar{U}}U^2 + (\frac{1}{6}Ae^{4\bar{U}} + Be^{2\bar{U}} + Ce^{\bar{U}})U \\
\qquad + \frac{1}{6}A(\frac{1}{12}A^2e^{6\bar{U}} + Be^{4\bar{U}} + Ce^{3\bar{U}}) + A^{-1}B(\frac{1}{2}Be^{2\bar{U}} + Ce^{\bar{U}}) \\
\tilde{G} = Ae^{2\bar{U}}U + \frac{1}{6}A^2e^{4\bar{U}} + Be^{2\bar{U}} + Ce^{\bar{U}}
\end{cases} \tag{82}
$$

$$
\begin{cases}
\Phi = \frac{1}{9}\bar{U}^{-2}U^3 + \frac{2}{3}(A\bar{U}^{-1/2} + B\bar{U}^{-3/2})U^{3/2} + A^2\bar{U} + B^2\bar{U}^{-1} \\
\tilde{G} = \frac{1}{3}\bar{U}^{-2}U^2 + (A\bar{U}^{-1/2} + B\bar{U}^{-3/2})U^{1/2}
\end{cases} \tag{83}
$$

According to [9], $\tilde{G}_{,UU} = 0$ is up to $U \leftrightarrow \bar{U}$ the standardized integrability condition for the existence of a second rank Killing tensor with a null eigenvector. Hence the two solutions (81) and (82) correspond to superintegrable geometries, admitting both second and third rank Killing tensors.

Type IIB

For this class of third rank Killing tensors it is possible to write down the implicit general solution to the quasi-linear, first order condition (48) as

$$F(\xi, \eta) = 0, \tag{84}$$

where F is an arbitrary function of its two arguments $\xi := U - \tilde{G}^2 \bar{U}$ and $\eta := \tilde{G}$. Symmetries ensure that if $\tilde{G} = f(U, \bar{U})$ is a solution, then so is $\tilde{G} = f(U + U_0, \bar{U} + \bar{U}_0)$, $\tilde{G} = \sqrt{b/a} f(aU, b\bar{U})$ and $\tilde{G} = [f(\bar{U}, U)]^{-1}$. A few explicit solutions can be obtained by choosing F such that eq. (84) becomes a polynomial equation in $\tilde{G}$ of sufficiently low order. The simplest nontrivial example of such a solution is obtained by setting $F(\xi, \eta) = \xi - 2\eta$. Solving the corresponding second order equation in $\tilde{G}$ yields

$$\tilde{G} = \left(1 \pm \sqrt{1 + U\bar{U}}\right) \bar{U}^{-1}. \tag{85}$$

4 Geometries admitting a homothetic vector field

In this section we consider the class of (1+1)-dimensional geometries that admit a homothetic vector field ξ, satisfying $\mathcal{L}_\xi g_{\alpha\beta} = 2g_{\alpha\beta}$. In the physical applications we have in mind the homothetic vector is timelike and we therefore restrict attention to this case. In fact, this implies no loss of generality since the timelike and spacelike cases are mathematically equivalent and the lightlike case is uninteresting as it requires a flat geometry. Adapting the coordinates to ξ, the metric can be written in the form

$$ds^2 = 2e^{2t} F(x)(-dt^2 + dx^2), \tag{86}$$

with $\xi = \partial/\partial t$. Referring to (5) and introducing the null variables $u = t + x$, $\bar{u} = t - x$ we see that $G = e^{2t} F(x)$ and that the corresponding null frame Ω^A is given by $\Omega^0 = e^t F^{1/2}(dt + dx)$, $\Omega^1 = e^t F^{1/2}(dt - dx)$. We assume that the Killing tensor, like the metric, has the rescaling property

$$\mathcal{L}_\xi K_{\alpha\beta\gamma} = 2b K_{\alpha\beta\gamma}, \tag{87}$$

where b is a constant whose value gives the weight $2b$ of the Killing tensor (cf. [9]). At least in the case in which the metric admits no Killing vector one can show that this is not a restriction (cf. [3]). The Killing tensor can then be factorized as

$$K_{\alpha\beta\gamma} = e^{2bt} \hat{K}_{\alpha\beta\gamma}, \quad \mathcal{L}_\xi \hat{K}_{\alpha\beta\gamma} = 0. \tag{88}$$

The null variables u, $\bar{u}$ will in general not be the Killing tensor adapted null variables U and $\bar{U}$, so from the outset the functions $S(u)$ and $R(\bar{u})$, as well as the functions $Q(\bar{u})$ and $P(\bar{u})$ introduced for type IIA and IIB respectively, has to be assumed arbitrary rather than taking the standard values 0 or 1. However, it follows from (88) that S and R must have the exponential dependence

$$S(u) = S_0 e^{2(b-3)u} = S_0 e^{2(b-3)(t+x)}, \quad R(\bar{u}) = R_0 e^{2(b-3)\bar{u}} = R_0 e^{2(b-3)(t-x)}, \tag{89}$$

and that the trace vector potential Φ up to an irrelevant additive constant must be of the form

$$\Phi = e^{2(b-1)t} \phi(x). \tag{90}$$

Substituting this into eq. (7), using $K_u = 2\Phi_{,u} = \Phi_{,t} + \Phi_{,x}$, $K_{\bar{u}} = -2\Phi_{,\bar{u}} = -\Phi_{,t} + \Phi_{,x}$, thus gives us the following general parametrization of $\hat{K}_{\alpha\beta\gamma}$:

$$
\begin{aligned}
\hat{K}_{000} &= -e^{-3t} R_0 e^{-2(b-3)x} F^{3/2} \\
\hat{K}_{111} &= -e^{-3t} S_0 e^{2(b-3)x} F^{3/2} \\
\hat{K}_{001} &= -e^{-3t} \tfrac{1}{2}[\phi' + 2(b-1)\phi] F^{-1/2} \\
\hat{K}_{011} &= -e^{-3t} \tfrac{1}{2}[\phi' - 2(b-1)\phi] F^{-1/2}.
\end{aligned}
\tag{91}
$$

13

At this point it is in place to note that the corresponding parametrization in the second rank case in [9] contains an error. The components of $\hat{K}_{MN}$ given in eq. (A4) are all missing a factor e^{-2t}. The consequence is that if $2b$ is to be interpreted as the weight of the Killing tensor $K_{MN} = e^{2bt}\hat{K}_{MN}$, one must in what follows eq. (A4) substitute b by $b - 1$. Therefore the error affects only the interpretation of b, not the results in [9]. Depending on the type of the Killing tensor, we now proceed as follows.

Type I

Here eq. (28) together with $G = \mathcal{K}_{,u\bar{u}}$ shows that one can assume without loss of generality that $\mathcal{K}$ is of the form

$$\mathcal{K} = e^{2t}k(x). \tag{92}$$

Hence by substituting this into eq. (29) one obtains a third order nonlinear ODE for the function $k(x)$ as the final condition. In terms of the functions

$$\Gamma = \frac{1}{2}(S_0 e^{2(b-3)x} + R_0 e^{-2(b-3)x}), \quad \Sigma = \frac{1}{2}(S_0 e^{2(b-3)x} - R_0 e^{-2(b-3)x}), \tag{93}$$

this condition reads

$$[3/4\Sigma\, k'' + b/2\Gamma\, k' + (b-3)\Sigma\, k]k''' + (2b-3)\Gamma\, (k'')^2 \tag{94}$$
$$+ (2b^2 - 3b - 3)\Sigma\, k'k'' + 2(2b-3)(b-4)\Gamma\, kk'' - 2b\Gamma\, (k')^2$$
$$+ 4(-2b^2 + 2b + 3)\Sigma\, kk' - 8(2b-3)(b-2)\Gamma\, k^2 = 0,$$

Note that when $b \neq 3$, there is no loss of generality in assuming $|R_0| = |S_0| = 1$ since one can make the translations $t \to t + t_0$, $x \to x + x_0$ with $t_0 = -\frac{1}{4(b-3)}\ln|R_0 S_0|$, $x_0 = \frac{1}{4(b-3)}\ln|R_0/S_0|$, under which $S_0 \to \operatorname{sgn}(S_0)$, $R_0 \to \operatorname{sgn}(R_0)$. With $b = -12/7$ and $R_0 = -S_0$, an example of a nontrivial solution to eq. (94) reads

$$\begin{cases} k(x) = \frac{49}{4}\sinh(6x/7)[\cosh(6x/7)]^{4/3} \\ F(x) = \sinh(6x/7)[\cosh(6x/7)]^{-2/3} \end{cases} \tag{95}$$

Type II

As before we shall assume that $R(\bar{u}) = 0$, thus letting the case $S(u) = 0$ be obtained via the transformation $u \leftrightarrow \bar{u}$ which here is equivalent to $x \to -x$.

Type IIA

Here the relation $\Phi_{,u} = Q(\bar{u})G$ implies that $Q(\bar{u}) = Q_0 e^{2(b-2)\bar{u}} = Q_0 e^{2(b-2)(t-x)}$. When setting $\phi(x) = S_0^{-1}Q_0^2 e^{-2(3b-7)x}\psi(x)$ and substituting eq. (90) into eq. (37), one obtains the condition

$$[1/16(\psi')^2 - 1/2(b-3)\psi\psi' + (b-3)^2\psi^2 - 1/2(b-1)^2\psi]\psi'' - 5/12(b-3)(\psi')^3 \tag{96}$$
$$+ [4(b-3)^2\psi + 1/2(2b-3)^2](\psi')^2 - [12(b-3)^3\psi^2 + 2(b-2)(3b-7)^2\psi]\psi'$$
$$+ 32/3(b-3)^4\psi^3 + 16(b-3)(b-2)^2\psi^2 = 0.$$

To standardize the Killing tensor, one can e.g. set $S_0 = 1$ while letting Q_0 determine the overall factor of $F(x)$. An example of a nontrivial solution to eq. (96) using this standardization is given by $b = 7/4$, $S_0 = 1$, $Q_0 = 25/3$ and

$$\begin{cases} \psi(x) = \frac{6}{125}(1 + e^{-5x/3})^3 \\ F(x) = e^{2x}(e^{3x/2} + e^{-x/6})^2 \end{cases} \tag{97}$$

14

Type IIB

Here $\Phi_{,\bar{u}} = P(\bar{u})$ implies that $P(\bar{u}) = P_0 e^{2(b-1)\bar{u}} = P_0 e^{2(b-1)(t-x)}$. Setting $F(x) = S_0^{-1/2} P_0^{1/2} e^{-2(b-2)x} H(x)$ and substituting $G = e^{2t} F(x)$ into eq. (49), one obtains the condition

$$-1/2(H^2 - 1)H' + 1/3(b-3)H^3 + (b-1)H = 0. \tag{98}$$

Analogous to type IIA, one can standardize the Killing tensor by setting $S_0 = 1$ while letting the value of P_0 determine the overall factor of $F(x)$. The general solution to eq. (98) can for all values of b be written down implicitly and for several values of b it is possible solve the algebraic equation for $H(x)$. Here we merely give the simplest nontrivial solution, for which $b = 0$, $S_0 = 1$, $P_0 = 1$ and

$$\begin{cases} H(x) = e^{-2x} \pm \sqrt{e^{-4x} - 1} \\ F(x) = e^{2x}(1 \pm \sqrt{1 - e^{4x}}). \end{cases} \tag{99}$$

Metrics with two exponential terms

Of special interest is the physically relevant case where $F(x)$ is of the form [13]

$$F(x) = C_1 e^{2mx} + C_2 e^{2nx}. \tag{100}$$

Making this ansatz and working through the integrability conditions for all three types of third rank Killing tensors yields five different solutions for which $m \neq n$ which are given in table 2. Unfortunately, only the

	m	n	b	Killing tensor type
(i)	3	1/3	1	IIA
(ii)	2	1/2	3/2	IIA
(iii)	3/5	−1/5	9/5	IIA
(iv)	1/3	−1/3	2	I, IIA
(v)	$i\sqrt{3}$	$-i\sqrt{3}$	3	I

(101)

Table 2: Geometries of the exponential type $ds^2 = 2e^{2t}(C_1 e^{2mx} + C_2 e^{2nx})(-dt^2 + dx^2)$ admitting a third rank Killing tensor.

trigonometric case (v) defines a new integrable geometry, as the geometries corresponding to the cases (i)-(iv) also admit at least one second rank Killing tensor [9]. In the cases (ii)-(iv), the existence of a third rank Killing tensor, namely the Nijenhuis bracket [12] of two independent second rank Killing tensors, could actually have been predicted from the outset. In case (iv) the geometry has a non-null Killing vector, so in this case there are a number of ways to construct a reducible third rank Killing tensor, which in table 2 is reflected by the fact that the Killing tensor type can be both I and IIA.

5   Concluding remarks

We have shown that the classification of third rank (1+1)-dimensional Killing tensors given in this paper can be used to find new explicit integrable geometries. Some examples of such geometries were given and many more can be constructed by using our results. Possible applications include inflationary models with a scalar field, anisotropic cosmologies and stellar models. In all of these cases the field equations can be formulated as geodesic equations on a (1+1)-dimensional geometry [13].

Unlike the case of second rank Killing tensors the separation of the geodesic equations for the third rank case cannot be done by a coordinate transformation on the configuration space. Instead it is necessary to apply a separating transformation which involves the entire phase space in a nontrivial way. The theory of such transformations is not fully understood. However, there does exist a recipe for finding separating variables [11].

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